DELEGATION APPROACH TO MONOTONE PERSUASION

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Abstract. We study a monotone persuasion problem. This is a problem of Bayesian persuasion between a principal and an agent, in which the principal's choice of information disclosure is restricted to monotone partitions. We show that this problem is equivalent to a constrained delegation problem, with the implication that solving one problem also means solving the other. We use this equivalence to apply known techniques in the delegation literature to address the monotone persuasion problem.

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1. Introduction

A monotone persuasion problem is a Bayesian persuasion problem of Kamenica and Gentzkow (2011), with additional structural assumptions. There are a principal (she) and an agent (he). The agent must choose an action, and his preferred action depends on a state, whereas the principal is has biased preferences relative to the agent. Ex-ante, the principal and agent are uninformed about the state and have a common prior. Before the agent's move, the principal designs an experiment (i.e., an information disclosure mechanism) that generates an informative message about the realized state. The agent observes the message, forms a posterior about the state, and then chooses the optimal action given the posterior. By designing an experiment, the principal manipulates the agent's posteriors about the state to further her own goals.

We assume the state and action spaces are intervals and that the principal's choice is restricted to monotone partitional experiments. These experiments produce messages that are deterministic and monotone in the state. That is, higher states always result in weakly higher messages. A simple example is an experiment that divides of the state space into intervals and informs the receiver about the interval containing the realized state.

Monotone partitional experiments constitute an important subclass of information disclosure mechanisms. Credit rating of financial institutions, consumer rating of services on AirBnB, Tripadvisor, Uber, etc., hygiene certification of restaurants, and grade conversion schemes from 100-point to ABC scale in schools and universities, are examples of such experiments. There are two defining features of monotone partitional experiments, *determinism* and *monotonicity*.

The first feature means that states are mapped into messages deterministically. This is a plausible constraints in many applications. A deliberate addition of randomness into a mechanism is hard to justify in practice and hard to enforce. Consider, for example, financial audit regulations that prescribe to invoke randomization when assigning a credit rating to a financial institution. This mechanism gives the power to the auditor to extract rents from the financial institution under scrutiny, by offering a greater rating for a reward. Whether or not the auditor does the randomization correctly is unverifiable. It can only be verified statistically, but it does not prevent the auditor to give favors occasionally, while maintaining the statistical average within an acceptable margin of error.

The second feature, monotonicity, means that if the state goes up, the resulting message never goes down. Given that states are totally ordered, if one accepts determinism of experiments, it is also very plausible to accept monotonicity. The monotonicity is a standard feature of practical mechanisms as well. For example, a better audit result of a financial institution never translates into a worse credit rating.

The central result of that paper that the monotone persuasion problem is equivalent to a *constrained delegation problem*, in the sense that in both problems the principal maximizes identical functionals. The implication of the equivalence that each solution of one problem can be translated into a solution of the other problem.

A constrained delegation problem describes an interaction between an uniformed principal and an informed agent, in which the agent must choose an action, and the principal can restrict the set of actions available to the agent. The word *constrained* refers to our assumption that the principal, when choosing an action set for the agent, cannot exclude the two extreme actions on the opposite ends of the action interval. This is an unusual assumption in the delegation literature, such as Alonso and Matouschek (2008) and Amador and Bagwell (2013). There are a few possible interpretations. One interpretation is that, ex ante, the agent is only aware of extreme decisions, and the principal can make the agent aware of some other decisions. Another interpretation is that extreme decisions are available by law and cannot be blocked by the principal. A related interpretation is that there exists a contractual relationship between the involved parties, so that the agent can always keep the contract unchanged or terminate the contract, but any other alterations must be permitted by the principal. For example, an employee (agent) wishes to reduce his job engagement to part time. An employer (principal) has the power to permit or prohibit any choice of the employee, except for the two extreme decisions of no action (remaining at full time job) and quitting the job entirely.

The common between the monotone persuasion and constrained delegation problems is that both are principal-agent models where an uniformed principal maximizes some objective by influencing actions of a (partially) informed agent. However, the mechanisms of influence are different. In the constrained delegation model, the principal directly restricts the set of allowed actions, whereas in the monotone persuasion model, the principal manipulates the agent's beliefs about the state. As we demonstrate by example in Section 3.2, this difference is substantive, and one problem cannot be mapped into another simply by equating the best-response actions of the agent.

To prove the equivalence result, we show that the monotone persuasion problem and the constrained delegation problem are both equivalent to a third problem. This is the problem of persuasion with binary actions and a privately informed agent, as in Kolotilin (2017) and Kolotilin et al. (2017). In this model, the agent chooses one of two actions. His preferred action depends on an unobservable state and his private type. Other than the above, the problem is identical to the monotone persuasion problem. The principal designs a monotone partitional experiment. The agent observes his type and observes a message from the experiment, and forms a posterior about the state. Then, the agent then chooses the optimal action given his type and the posterior.

First, to show the equivalence of this problem to the monotone persuasion problem, we simply change the order of events. First, the principal designs a monotone partitional experiment. Second, the agent observes a message from the experiment and forms a posterior about the state. Third, the agent chooses a strategy that describes what action will be taken as a function of each possible type. Finally, the agent observes his type and carries out the strategy chosen earlier. In this interpretation, the agent makes a decision while being uninformed about his type. Under appropriate single-crossing assumptions of the agent's preferences, the agent's optimal strategy is a threshold type, such that agent chooses one action whenever the realized type is above the threshold and the other action otherwise.. That is, the agent has no private information and chooses a decision from an interval (of types). This is a monotone persuasion problem.

Second, to show the equivalence of the above problem to the constrained delegation problem, again, we change the order of events. First, the principal designs a monotone partitional experiment. Second, the agent observes his type. Third, the agent chooses a strategy that describes what action will be taken as a function of each possible message of the experiment. Finally, the agent observes a message from the experiment and carries out the strategy chosen earlier. In this interpretation, the agent makes a decision while being fully informed about his type. Under appropriate single-crossing assumptions of the agent's preferences, the agent's optimal strategy is a threshold message, such that agent chooses one action whenever the realized message is below the threshold and the other action otherwise. That is, the informed agent is restricted to choose a threshold message from the set of messages that the experiment can generate. As the set of experiment's messages is designed by the principal, this can be interpreted as a delegation problem. Notice that the agent can always have a choice of ignoring the message and always choose one (or the other) action. This is as if the threshold is so extreme that realized messages are below (or above) the threshold with probability one. These extreme choices of thresholds must be available to the agent and cannot be removed by the principal.

The equivalence between these three models can be used to use the known methodology and solution techniques for one problem to solve other problems. We use this equivalence to connect the interval delegation results for the delegation model and the interval revelation and the upper/lower-censorship results for the persuasion models.

In addition, we consider the subclasses of these models, with an extra assumption of payoff linearity in the state/type, are also equivalent. In the persuasion model, we assume that the sender's and receiver's payoffs are linear in the state, as in Gentzkow and Kamenica (2016) and Kolotilin et al. (2017). In the delegation model, we assume that the principal's and agent's payoffs are linear in the type, as in Alonso and Matouschek (2008) and Amador and Bagwell (2013).[1] We show that these models with linear payoffs are also equivalent.

[1]In Alonso and Matouschek (2008), the payoffs are quadratic. But they can be written as linear in the type without loss of generality, as the quadratic term of the type does not interact with the action, and thus can be omitted.

2. Two Problems

2.1. **A Problem.** There are a principal (she) and an agent (he). The agent chooses a decision y from the interval $[0,1]$. The principal's and agent's payoffs

$$V(\omega, y) \text{ and } U(\omega, y)$$

depend on the chosen decision y and on a payoff-relevant state $\omega \in [0,1]$. The state is a random variable with a commonly known distribution F. We assume that[2]

(A_0) F admits a strictly positive density;

(A_1) $U(\omega, y)$ and $V(\omega, y)$ are continuous in ω and continuously differentiable in y;

(A_2) $\frac{\partial}{\partial y}U(\omega, y)$ is strictly increasing in ω and strictly decreasing in y.

Under these assumptions, without loss of generality we can redefine the state such that its distribution is uniform.[3] Thus, in what follows we shall assume

(A_3) F is uniform on $[0,1]$.

A pair (U, V) is called a *primitive* of the problem. Denote by $\mathcal{P}$ the set of all primitives that satisfy the above assumptions (A_1)–(A_2).

We now describe two variants of this problem that differ only in the tools that the principal uses to affect the agent's decisions.

2.2. **Monotone Persuasion Problem.** In a monotone persuasion problem, the principal manipulates the agent's beliefs about the state via an information disclosure mechanism. The principal designs a Blackwell experiment that sends to the agent (stochastic) messages which depend on a realized state. In this paper, we restrict attention to monotone partitional experiments whose recommendations are nonrandom and nondecreasing in the state.

Formally, a *monotone partitional experiment* is a nondecreasing function $\pi : [0,1] \to \mathbb{R}$ that maps each state ω into a message $\pi(\omega)$. These experiments generate a monotone partition of the state space and inform the agent about the element of the partition that the realized state belongs to. Note that a monotone partition need not be finite, there can be separating intervals where the state is precisely disclosed to the agent.

A monotone partitional experiment π is *diagonal* if

$$\begin{aligned} &\pi(\omega) = \inf\{t : \pi(t) = \pi(\omega)\} \quad \text{for all } \omega \in [0,1), \\ &\pi(1) = 1. \end{aligned} \tag{1}$$

[2]In Section 5 we discuss how these assumptions can be relaxed without affecting our results.

[3]Suppose that (A_0)–(A_2) hold. Define $\tilde{\omega} = F(\omega)$. Since F^{-1} is continuous and strictly increasing by (A_0), $\tilde{U}(\tilde{\omega}, y) = U(F^{-1}(\tilde{\omega}), y)$ and $\tilde{V}(\tilde{\omega}, y) = U(F^{-1}(\tilde{\omega}), y)$ are continuous in ω, and $\frac{\partial}{\partial y}\tilde{U}(\tilde{\omega}, y) = \frac{\partial}{\partial y}U(F^{-1}(\tilde{\omega}), y)$ satisfies (A_2).

Thus, each element $[x', x'')$ of an induced partition is identified with its lowest point x', so that whenever $\omega \in [x', x'')$, the experiment sends the message $\pi(\omega) = x'$. In what follows, without loss of generality we restrict attention to diagonal experiments, and denote the set of such experiments by Π^*.[4]

Consider a primitive $(U_P, V_P) \in \mathcal{P}$, where we use subscript P to refer to the persuasion problem. Given a message m of an experiment π, the agent chooses a decision that maximizes his expected payoff w.r.t. the posterior belief induced by that message,

$$y^*_\pi(m) \in \arg\max_{y\in[0,1]} \mathbb{E}[U_P(\omega, y) \mid \pi(\omega) = m],$$

and $y^*_\pi(m)$ is arbitrary if message m is not in the image of π. The principal's objective is to choose an experiment $\pi \in \Pi^*$ to maximize her expected payoff, subject to the agent behaving optimally,

$$\max_{\pi\in\Pi^*} \mathbb{E}[V_P(\omega, y^*_\pi(\pi(\omega)))]. \tag{2}$$

2.3. **Constrained Delegation Problem.** Let us now describe a constrained delegation problem in which the agent is informed about the state, and the principal affects the agent's choice by restricting the set of agent's decisions.

Formally, consider a primitive $(U_D, V_D) \in \mathcal{P}$, where we use subscript D to refer to the delegation problem. The principal chooses a subset X of the agent's decisions called a *delegation set*, where X is a closed subset of $[0, 1]$. Then, state $\theta \in [0, 1]$ realises. (We use a different notation for the state to differentiate it from the state in the persuasion problem.) The agent observes θ and chooses a decision from the delegation set X,

$$y^*_X(\theta) \in \arg\max_{y\in X} U_D(\theta, y).$$

In the constrained delegation problem, the principal can allow or prohibit the agent to make any decisions, but the very extremes, 0 and 1, must be allowed. A set X that respects this assumption,

$$\begin{aligned} &X \subset [0, 1] \text{ is closed}, \\ &\{0, 1\} \subset X. \end{aligned} \tag{3}$$

will be called a *constrained delegation set*. Denote by $\mathcal{X}^*$ the set of such constrained delegation sets.

The principal chooses a constrained delegation set $X \in \mathcal{X}^*$ to maximize her expected payoff

$$\max_{X\in\mathcal{X}^*} \mathbb{E}\big[V_D(\theta, y^*_X(\theta))\big]. \tag{4}$$

[4]By definition, a diagonal experiment π is right-continuous. Therefore, each non-singleton element of a partition generated by π is a half-open interval $[x', x'')$. Since the distribution of the state is nonatomic, for each monotone partition of the state space, each interval element (x', x''), $[x', x'']$, or $(x', x'']$ can be replaced by $[x', x'')$ without changing the payoffs. A partition whose elements are singletons and left-closed, right-open intervals can be induced by a diagonal experiment.

3. Equivalence

3.1. **Definition.** Two problems are equivalent if there is an operational way to associate each primitive of one problem with a primitive of another, and each principal's strategy in one problem with a principal's strategy in another, such that the ex-ante expected payoff of the principal is preserved.

In order to connect the monotone persuasion problem to the constrained delegation problem, we need some notations.

Consider a one-to-one mapping $\mu : \Pi^* \to \mathcal{X}^*$ that associates each monotone partitional experiment π with a unique constrained delegation set $X = \mu(\pi)$. We say that primitives $(U_P, V_P) \in \mathcal{P}$ and $(U_D, V_D) \in \mathcal{P}$ of the monotone persuasion and constrained delegation problems are *equivalent under* μ, denoted by

$$(U_P, V_P) \sim_\mu (U_D, V_D),$$

if there exists a constant C such that

$$\mathbb{E}\big[V_P(\omega, y^*_\pi(\omega))\big] = \mathbb{E}\big[V_D(\theta, y^*_{\mu(\pi)}(\theta))\big] + C \quad \text{for all } \pi \in \Pi^*. \tag{5}$$

In words, the principal's preferences are identical for each monotone partitional experiment π and each constrained delegation set X that satisfies $X = \mu(\pi)$. Importantly, if $(U_P, V_P) \sim_\mu (U_D, V_D)$, then the principal's optimal solution is the same in both problems,

$$\arg\max_{\pi \in \Pi^*} \mathbb{E}\big[V_P(\omega, y^*_\pi(\omega))\big] = \arg\max_{X \in \mathcal{X}^*} \mathbb{E}\big[V_D(\theta, y^*_X(\theta))\big].$$

We say that the monotone persuasion problem and the constrained delegation problem are *equivalent* if there exists a one-to-one mapping μ such that, for each primitive in $\mathcal{P}$ of one problem there exists an equivalent-under-μ primitive in $\mathcal{P}$ of another problem.

3.2. **Naive approach.** Before we state our main result, let us discuss why a naive way of associating persuasion and delegation does not provide equivalence.

Consider the following way of associating a monotone persuasion problem with a delegation problem that does not necessarily have the constraint $\{0, 1\} \in X$. For a given primitive (U, V) of the monotone persuasion problem and a given experiment π, each message m of this experiment induces a unique best-response action $y^*_\pi(m)$ of the agent. Thus, the set of messages $\pi([0, 1])$ defines a set of best-response actions $Y_\pi \subset [0, 1]$. Consider now a delegation problem with the same primitive, (U, V), in which the agent is informed about the state and is restricted to choose an action from set Y_π. We argue that agent's optimal choices and, hence, the principal's payoffs, will generally differ in these two problems.

For example, let $(U, V) \in \mathcal{P}$ satisfy $U(\omega, y) = -(\omega - y)^2$. Consider the experiment π^* that informs the agent whether the state is above or below 0.4. Since the agent minimizes the distance to the posterior mean of the state, the induced best-response action of the agent is

$$y^*_{\pi^*}(\pi^*(\omega)) = \begin{cases} 0.2, & \omega < 0.4, \\ 0.7, & \omega \geq 0.4. \end{cases}$$

Consider now the delegation problem with the same primitive, and let the delegation set be $X^* = \{0.2, 0.7\}$. Since the agent observes the state and minimizes the distance to it, the induced best-response action of the agent is

$$y^*_{X^*}(\omega) = \begin{cases} 0.2, & \omega < 0.45, \\ 0.7, & \omega \geq 0.45, \end{cases}$$

where 0.45 is the midpoint between 0.2 and 0.7. Note that if, for example,

$$V(\omega, y) = \begin{cases} -(0.2 - y)^2(0.4 - \omega)^2, & \omega < 0.4, \\ -(0.7 - y)^2(0.4 - \omega)^2, & \omega \geq 0.4, \end{cases}$$

then experiment π^* achieves the principal's first best in the persuasion problem (U, V), but this first-best payoff is unattainable by any delegation set $X \subset [0, 1]$ in the delegation problem with the same primitive. Conversely, if

$$V(\omega, y) = \begin{cases} -(0.2 - y)^2(0.45 - \omega)^2, & \omega < 0.45, \\ -(0.7 - y)^2(0.45 - \omega)^2, & \omega \geq 0.45, \end{cases}$$

then delegation set X^* achieves the principal's first best in the delegation problem (U, V), but this first-best payoff is unattainable by any monotone experiment $\pi \in \Pi^*$ in the persuasion problem with the same primitive.

3.3. **Main Result.** Let us state our main result.

Theorem 1. *Let μ associate each monotone partition experiment $\pi \in \Pi^*$ with the constrained delegation set equal to the image of π,*

$$\mu(\pi) = \pi([0, 1]). \tag{6}$$

Consider primitives $(U_P, V_P) \in \mathcal{P}$ and $(U_D, V_D) \in \mathcal{P}$ of the monotone persuasion and constrained delegation problems. If, for all $(\omega, \theta) \in [0, 1]^2$,

$$\frac{\partial}{\partial \theta} U_P(\omega, \theta) + \frac{\partial}{\partial \omega} U_D(\theta, \omega) = 0, \quad \textit{and} \tag{7}$$

$$\frac{\partial}{\partial \theta} V_P(\omega, \theta) + \frac{\partial}{\partial \omega} U_D(\theta, \omega) = 0, \tag{8}$$

then $(U_P, V_P) \sim_\mu (U_D, V_D)$.

For a given primitive of one problem, we can use Theorem 1 to construct an equivalent primitive of the other problem. For example, let (U_P, V_P) be a primitive of the monotone persuasion problem. The equivalent-under-μ primitive (U_D, V_D) is defined as follows. For each $\theta \in [0,1]$ and each $y \in [0,1]$ let

$$U_D(\theta, y) = -\int_0^y \frac{\partial}{\partial \theta} U_P(\omega, \theta) \mathrm{d}\omega, \quad \text{and}$$
$$V_D(\theta, y) = -\int_0^y \frac{\partial}{\partial \theta} V_P(\omega, \theta) \mathrm{d}\omega.$$

The next corollary is immediate.

Corollary 1. *The monotone persuasion problem and the constrained delegation problem are equivalent.*

3.4. **Persuasion with Posterior Mean Dependent Payoffs.** In this section, we consider the monotone persuasion problem where the principal's payoff depends on the posterior beliefs about the state only through the posterior mean state, assuming the agent chooses an optimal action under this posterior. This setting, albeit without the restriction to monotone experiments, has been studied in Gentzkow and Kamenica (2016), Dworczak and Martini (2017), Kolotilin (2017), and Kolotilin et al. (2017).

As argued in Gentzkow and Kamenica (2016), without loss of generality we can assume that the agent's payoff is

$$U_P(t, y) = -\tfrac{1}{2}(t - y)^2$$

and the principal's payoff is an arbitrary function of the agent's action,

$$V_P(t, y) = \nu(y),$$

where $t \in [0,1]$ the state distributed according to some distribution F. We assume that U_P, V_P, and F satisfy assumptions (A_0)–(A_2). Redefining the state $\omega = F(t)$ and using the notation $b(\omega) = F^{-1}(\omega)$, we have

$$U_P(\omega, y) = -\tfrac{1}{2}(b(\omega) - y)^2 \quad \text{and} \quad V_P(\omega, y) = \nu(y),$$

and $(U_P, V_P) \in \mathcal{P}$.

Let us use Theorem 1 to derive an equivalent constrained delegation problem. Consider a primitive (U_P, V_P) of the monotone persuasion problem as described above. We now find U_D and V_D that satisfy (7)–(8).

For each $\theta \in [0,1]$ and each $y \in [0,1]$, define

$$U_D(\theta, y) = -\int_0^y \frac{\partial}{\partial \theta} U_P(\omega, \theta) \mathrm{d}\omega = -\int_0^y (b(\omega) - \theta) \mathrm{d}\omega = \theta y - \int_0^y b(\omega) \mathrm{d}\omega,$$

and

$$V_D(\theta, y) = -\int_0^y \frac{\partial}{\partial \theta} V_P(\omega, \theta) \mathrm{d}\omega = -\int_0^y \nu'(\theta) \mathrm{d}\omega = -\nu'(\theta) y.$$

The pair $(U_D, V_D) \in \mathcal{P}$ is a primitive of the constrained delegation problem.

Finally, to represent the above payoffs as in Proposition 2 of Amador and Bagwell (2013), denote $B(y) = -\int_0^y b(\omega)\mathrm{d}\omega$ and $D(\theta) = -\nu'(\theta)$. Then,

$$U_D(\theta, y) = \theta y + B(y) \quad \text{and} \quad V_D(\theta, y) = D(\theta)y.$$

4. Proof of Theorem 1

4.1. **Equivalence of the principal's choice sets.** Any experiment $\pi \in \Pi^*$ can be equivalently described by its image, the set of π's messages $X = \pi([0,1])$. Note that, because π is right-continuous and diagonal, its image X is a closed subset of $[0,1]$ and contains the extreme messages 0 and 1, i.e., $X \in \mathcal{X}^*$. Conversely, any constrained delegation set $X \in \mathcal{X}^*$ can be equivalently described by a diagonal experiment π defined by

$$\pi(\omega) = \sup\{x \in X : x \leq \omega\}.$$

Note that, because X is closed, π is right-continuous, and thus π is in Π^*. Consequently, the mapping $\mu : \Pi^* \to \mathcal{X}^*$ defined by (6) is one-to-one, as it uniquely associates each monotone experiment $\pi \in \Pi^*$ with a constrained delegation set $X \in \mathcal{X}^*$, and vice versa.

In what follows, we represent monotone experiments by the associated delegation sets, and consider $\mathcal{X}^*$ as the principal's choice set in both contexts, persuasion and delegation.

4.2. **Monotone persuasion problem with a privately informed agent.** To prove the equivalence of the monotone persuasion and constrained delegation problems under the above μ, we show that they are equivalent to a third problem. This is the problem of persuasion with binary actions and a privately informed receiver, as in Kolotilin (2017) and Kolotilin et al. (2017), which is described as follows.

There are a principal (she) and an agent (he). The agent decides between being *active* ($a = 1$) and *inactive* ($a = 0$). His preferred decision depends on an unobservable state $\omega \in [0,1]$ and his private type $\theta \in [0,1]$ that are independent random variables that are uniformly distributed on $[0,1]$.

If the agent is inactive, $a = 0$, then the payoffs of the agent and principal are normalized to zero. If the agent is active, $a = 1$, then the payoffs of the principal and agent are denoted by $v(\omega, \theta)$ and $u(\omega, \theta)$, respectively. We assume that:

(A_1') u and v are continuous;

(A_2') $u(\omega, \theta)$ is strictly increasing in ω and strictly decreasing in θ.

A pair (u, v) that satisfies the above assumptions is a primitive of this problem.

Before the agent makes his decision, he observes a message from a monotone experiment described by a set $X \in \mathcal{X}^*$. The timing of the interaction is as follows:

(i) the principal chooses an experiment described by its set of messages $X \in \mathcal{X}^*$;

(ii) the agent observes his type and a message from the experiment;

(iii) the agent chooses an action, 0 or 1.

Let π be the diagonal monotone experiment associated with X,

$$\pi(\omega) = \sup\{x \in X : x \leq \omega\}.$$

For each message $x \in X$ and each type $\theta \in [0, 1]$, let $\bar{u}_X(x, \theta)$ be the expected payoff of the agent of type θ if he receives message x and chooses $a = 1$,

$$\bar{u}_X(x, \theta) = \mathbb{E}\big[u(\omega, \theta)\big|\pi(\omega) = x\big].$$

4.3. **Equivalence to monotone persuasion.** Consider the following sequence of moves:

(i) the principal chooses an experiment described by a set $X \in \mathcal{X}^*$;

(ii) the agent observes a message x from the experiment;

(iii) the agent makes a decision, a strategy $a_x : [0, 1] \to \{0, 1\}$ that specifies the action $a_x(\theta)$ as a function of his type;

(iv) the agent observes his type θ and chooses the action $a_x(\theta)$ specified at stage (iii).

Fix $x \in X$. By (A$_2'$), there exists $y \in [0, 1]$ such that $\bar{u}(x, \theta) < (>) 0$ when $\theta > (<) y$. So the choice of the agent is reduced to a choice of threshold type $y \in [0, 1]$ such that $a = 1$ when $\theta < y$ and $a = 0$ when $\theta > y$. Under a choice of threshold $y \in [0, 1]$, the agent's expected payoff at stage (iii) is given by

$$\int_0^y \bar{u}(x, \theta)\mathrm{d}\theta = \int_0^y \mathbb{E}\big[u(\omega, \theta)\big|\pi(\omega) = x\big]\mathrm{d}\theta = \mathbb{E}\big[U_P(\omega, y)\big|\pi(\omega) = x\big],$$

where we denote

$$U_P(\omega, y) = \int_0^y u(\omega, \theta)\mathrm{d}\theta. \tag{9}$$

Thus, after observing a message $x \in X$, the agent's objective is

$$\max_{y \in [0,1]} \mathbb{E}\big[U_P(\omega, y)\big|\pi(\omega) = x\big],$$

which is identical to the agent's objective in the monotone persuasion problem with the payoff function U_P given by (9).

Let $y^*_X(x)$ be the solution of the above agent's problem. Denote

$$V_P(\omega, y) = \int_0^y v(\omega, \theta)\mathrm{d}\theta. \tag{10}$$

The principal's ex-ante expected payoff is equal to

$$\int_0^1 \int_0^{y^*_X(\pi(\omega))} \mathbb{E}\big[v(t, \theta)\big|\pi(t) = \pi(\omega)\big]\mathrm{d}\theta\mathrm{d}\omega = \int_0^1 \mathbb{E}\big[V(t, y^*_X(\pi(\omega)))\big|\pi(t) = \pi(\omega)\big]\mathrm{d}\omega$$
$$= \mathbb{E}\big[V(\omega, y^*_X(\pi(\omega)))\big].$$

This is identical to the principal's expected payoff in the monotone persuasion problem with primitive (U_P, V_P) for each $X \in \mathcal{X}^*$, where U_P and V_P are given by (9)–(10). It is straightforward to verify that (U_P, V_P) satisfies assumptions (A_1)–(A_2) if (u, v) satisfies assumptions (A'_1)–(A'_2).

Conversely, let (U_P, V_P) be a primitive of the monotone persuasion problem in $\mathcal{P}$. Define

$$u(\omega, \theta) = \frac{\partial}{\partial\theta} U_P(\omega, \theta) \quad \text{and} \quad v(\omega, \theta) = \frac{\partial}{\partial\theta} V_P(\omega, \theta). \tag{11}$$

It is easy to verify that (u, v) is a primitive of the monotone persuasion problem with privately informed agent that satisfies (A'_1)–(A'_2), such that, for each $X \in \mathcal{X}^*$, the expected payoff of the principal is the same as that in (U_P, V_P) up to a constant independent of X.

4.4. **Equivalence to constrained delegation.** Consider the following sequence of moves:

(i) the principal chooses an experiment described by a set $X \in \mathcal{X}^*$;

(ii) the agent observes his type θ;

(iii) the agent makes a decision, a strategy $a_\theta : X \to \{0, 1\}$ that specifies the action $a_\theta(x)$ as a function of message x from the experiment;

(iv) the agent observes a message x from the experiment and chooses the action $a_\theta(x)$ specified at stage (iii).

Fix a type θ. By (A'_2), there exists $x' \in X$ such that $\bar{u}_X(x, \theta) > (<) 0$ when $x > (<) x'$. So the choice of the agent is reduced to a choice of threshold $y \in X$ such that $a = 1$ when $x \geq y$ and $a = 0$ when $x < y$. In particular, if $\bar{u}_X(x, \theta) > (<) 0$ for all $x \in X$, then the optimal threshold $y = 0$ ($y = 1$) corresponds to the decision of ignoring the message and always being active, $a = 1$ (respectively, inactive, $a = 0$) for all messages.

Under a choice of threshold $y \in X$, the agent's expected payoff at stage (iii) is given by

$$\int_{\omega:\pi(\omega)\geq y} \bar{u}_X(\pi(\omega), \theta)\mathrm{d}\omega = \int_y^1 \mathbb{E}\big[u(t, \theta)\big|\pi(t) = \pi(\omega)\big]\mathrm{d}\omega = U_D(\theta, y),$$

where we denote

$$U_D(\theta, y) = \int_y^1 u(\omega, \theta)\mathrm{d}\omega. \tag{12}$$

Thus, after observing a type $\theta \in [0, 1]$, the agent's objective is

$$\max_{y\in X} U_D(\theta, y),$$

which is identical to the agent's objective in the constrained delegation problem with the payoff function U_D given by (12).

Define

$$V_D(\theta, y) = \int_y^1 v(\omega, \theta)\mathrm{d}\omega. \tag{13}$$

Let $y^*_X(\theta)$ be the solution of the above agent's problem. The principal's ex-ante expected payoff is equal to

$$\int_0^1 \left(\int_{y^*_X(\theta)}^1 v(\omega, \theta)\mathrm{d}\omega \right) \mathrm{d}\theta = \int_0^1 V_D(\theta, y^*_X(\theta))\mathrm{d}\theta.$$

This is identical to the principal's expected payoff in the constrained delegation problem with primitive (U_D, V_D) for each $X \in \mathcal{X}^*$, where U_D and V_D are given by (12)–(13). It is straightforward to verify that (U_D, V_D) satisfies assumptions (A_1)–(A_2) if (u, v) satisfies assumptions (A'_1)–(A'_2).

Conversely, let (U_D, V_D) be a primitive of the constrained delegation problem in $\mathcal{P}$. Define

$$u(\omega, \theta) = -\frac{\partial}{\partial \omega} U_D(\theta, \omega) \quad \text{and} \quad v(\omega, \theta) = -\frac{\partial}{\partial \omega} V_D(\theta, \omega). \tag{14}$$

It is easy to verify that (u, v) is a primitive of the monotone persuasion problem with privately informed agent that satisfies (A'_1)–(A'_2) such that, for each $X \in \mathcal{X}^*$, the expected payoff of the principal is the same as that in (U_D, V_D) up to a constant independent of X.

Lastly, (7)–(8) immediately follow from (11) and (14).

5. Discussion of Assumptions

Our assumptions (A_0)–(A_2) are chosen for simplicity and clarity of exposition, and can be relaxed to an extent.

Strict single-crossing properties. Assumption (A_2) can be substantially relaxed. It suffices to require that the agent's utility satisfies specific strict single-crossing properties as described below, without need for any changes in the proof.

Before we proceed, we need some definitions. Consider a function $g(x_1, x_2) : [0, 1]^2 \to \mathbb{R}$. We say that:

(i) g satisfies the *strict single crossing property (SSC) from below* w.r.t. x_i if for each $x_{-i} \in [0, 1]$ there exists a threshold $\bar{x} \in [0, 1]$ such that

$$x_i > (<)\, \bar{x} \quad \Longrightarrow \quad g(x_i, x_{-i}) > (<)\, 0.$$

(ii) g satisfies the *strict aggregate single crossing property (SASC) from below* w.r.t. x_i if for each probability distribution H of x_{-i} there exists a threshold $\bar{x} \in [0, 1]$ such that

$$x_i > (<)\, \bar{x} \quad \Longrightarrow \quad \int_0^1 g(x_i, x_{-i})\mathrm{d}G(x_{-i}) > (<)\, 0.$$

(iii) g satisfies *strict single crossing of differences (SSCD) from below* w.r.t. x_i if for each $x'_{-i}, x''_{-i} \in [0,1]$, $x'_{-i} < x''_{-i}$, there exists a threshold $\bar{x} \in [0,1]$ such that

$$x_i > (<)\, \bar{x} \quad \Longrightarrow \quad g(x_i, x''_{-i}) - g(x_i, x'_{-i}) > (<)\, 0.$$

(iv) g satisfies SSC (SASC,SSCD) from *above* w.r.t. x_i if $-g$ satisfies SSC (SASC,SSCD) from below w.r.t. x_i.

Let $\bar{\mathcal{P}}$ be the set of primitives of the monotone persuasion problem that satisfy assumptions (A_1), (A_3), and the following weakening of (A_2):

(A_2^P) $\frac{\partial}{\partial y} U_P(\omega, y)$ is SSC from below in ω and SASC from above in y.

Let $\bar{\mathcal{D}}$ be the set of primitives of the constrained delegation problem that satisfy assumptions (A_1), (A_3), and the following weakening of (A_2):

(A_2^D) $U_D(\theta, y)$ is strictly quasiconcave in y and SSCD from below in θ.

Informally, assumption (A_2^P) requires that the agent has a unique best-response decision under any belief about the state ω, and that higher states induce higher best-response decisions. Assumption (A_2^D) requires that the agent has a unique best-response decision for any state θ (recall that the agent observes θ in the delegation problem), and that higher states induce higher best-response decisions. Notice that the condition that $U_D(\theta, y)$ is SSCD in θ can be equivalently defined as $\int_{y'}^{y''} \frac{\partial}{\partial \omega} U_D(\theta, \omega) \mathrm{d}\omega$ to be SSC in θ for all $y' < y''$. Thus, the condition that $U_D(\theta, y)$ is SSCD in θ is more demanding than $\frac{\partial}{\partial y} U_D(\theta, y)$ to be *SSC* in θ, but less demanding than $\frac{\partial}{\partial y} U_D(\theta, y)$ to be *SASC* in θ.

The statement of Theorem 1 can be expanded to these classes of primitives, that is, for all $(U_P, V_P) \in \bar{\mathcal{P}}$ and $(U_D, V_D) \in \bar{\mathcal{D}}$, if (7) and (8) hold for all $(\omega, \theta) \in [0,1]^2$, then $(U_P, V_P) \sim_\mu (U_D, V_D)$. The proof of Theorem 1 applies without changes, once we establish that U_P satisfying (A_2^P) and U_D satisfying (A_2^D) are sufficient[5] to guarantee that, for all $X \in \mathcal{X}^*$, the function

$$\begin{aligned} \bar{u}_X(x, \theta) &= \mathbb{E}\big[u(\omega, \theta) \big| \pi(\omega) = x\big] \\ &= \mathbb{E}\left[-\frac{\partial}{\partial \omega} U_D(\theta, \omega) \middle| \pi(\omega) = x \right] = \mathbb{E}\left[\frac{\partial}{\partial \theta} U_P(\omega, \theta) \middle| \pi(\omega) = x \right] \end{aligned}$$

satisfies SSC from above in $\theta \in [0,1]$ and SSC from below in $x \in X$

Multiplicity of the agent's best-response decisions.

Our assumptions ensure that the agent's best-response decision is always unique. However, as in Kamenica and Gentzkow (2011), we can assume that whenever the agent has multiple best-response decisions, he chooses the one that the principal

[5] Also necessary, as can be shown by adapting Theorem 1 of Quah and Strulovici (2012) to the notion of *strict* single crossing.

prefers (with ties broken arbitrarily both parties are indifferent). Then, from perspective of the principal, the best-response of the agent is payoff-unique, and Theorem 1 applies.

Thus, two assumption details whose sole purpose is to ensure uniqueness of the agent's best-response decision can be relaxed. Specifically, the above strict single-crossing conditions can be replaced by their "weak" versions, and condition (A_0), which implies that F^{-1} is strictly increasing, can be replaced by the condition that F is strictly increasing, i.e., we can allow for atoms in the distribution.

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